Sequences

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Find all real sequences $(a_n)_{n\geq 1}$ such that $a_1 = 1,\ a_{n+3} = 5a_{n+2} - 8a_{n+1} + 4a_n$ for every $n \geq 1$ and $\lim_{n\to\infty} \dfrac{a_n}{2^n} = 3.$

Solution by Arkady Alt , San Jose, California, USA.

Since $a_{n+3} = 5a_{n+2} - 8a_{n+1} + 4a_n \Leftrightarrow a_{n+3} - a_{n+2} - 4(a_{n+2} - a_{n+1}) + 4(a_{n+1} - a_n) = 0 \Leftrightarrow$ $\dfrac{a_{n+3} - a_{n+2}}{2^{n+1}} - 2 \cdot \dfrac{a_{n+2} - a_{n+1}}{2^n} + \dfrac{a_{n+1} - a_n}{2^{n-1}} = 0, n \in \mathbb{N}$ then $\dfrac{a_{n+1} - a_n}{2^{n-1}} = pn + q \Leftrightarrow$ $a_{n+1} - a_n = 2^{n-1}(pn + q)$ and, therefore, $a_n - a_1 = a_n - 1 = \sum_{k=1}^{n-1} 2^{k-1}(pk + q) =$ $p - q + 2^{n-1}(q - 2p) + 2^{n-1}np.$ Since $\lim_{n\to\infty} \dfrac{a_n}{2^n} = 3$ implies $p = 0$ and $\dfrac{q - 2p}{2} = 3$ then $q = 6$ and $a_n = 1 - 6 + 6 \cdot 2^{n-1} = 3 \cdot 2^n - 5.$